ON THE DAMPING OF THE ANGULAR MOMENTUM OF THREE HARMONIC OSCILLATORS

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ABSTRACT

In the frame of the Lindblad theory of open quantum systems, the system of three uncoupled harmonic oscillators with opening operators linear in the coordinates and momenta of the considered system is analyzed. The damping of the angular momentum and of its projection is obtained.

1. Introduction

In the last two decades, more and more interest arose about the problem of dissipation in quantum mechanics, i.e. the consistent description of open quantum systems [1-6].

Because dissipative processes imply irreversibility, and, therefore, a preferred direction in time, it is generally thought that quantum dynamical semigroups are the basic tools to introduce dissipation in quantum mechanics. The most general form of the generators of such semigroups (under some topological conditions) was given by Lindblad [7-9], whose formalism has recently been applied to various physical phenomena, for instance, to the damping of collective modes in deep inelastic collisions [10-15]. An important feature of these reactions is the dissipation of energy and angular momentum out of the collective degrees of freedom into the intrinsic or single-particle degrees of freedom.

The damping of energy and angular momentum can be described at different levels of approximations: one may use quantum or classical mechanical methods and microscopical or phenomenological models.

In the previous papers [13-15] simple phenomenological models for the damping of the angular momentum were studied in the framework of the Lindblad quantum mechanical theory. In the present paper, for a special choice of the opening coefficients we obtain explicit expressions for the damping of the angular momentum and projection of the angular momentum of three harmonic oscillators. As opening operators we use the generators of the Heisenberg group, the coordinates q_k and momenta $p_k (k = 1, 2, 3)$. In order to keep the symmetry of the system we assume that the opening coefficients are symmetric under the permutation of the axes. In this way we include the influence of the coupling, due to the environment, of the three harmonic oscillators, initially uncoupled.

The paper is organized as follows: In Sec.2 we first present the Lindblad equation of motion in the Heisenberg picture. In Sec.3 we obtain the damping of the projection of the angular momentum for a system of three uncoupled harmonic oscillators. In Sec.4 we analyze the behaviour of the squared angular momentum for the same system.

2. The Lindblad formalism in the Heisenberg picture

In Lindblad's formalism, the usual von Neumann-Liouville equations ruling the time evolution of closed quantum systems are replaced by the following ones [7-9]:

$$\frac{d\Phi_t(\rho)}{dt} = L[\Phi_t(\rho)],$$

$$\frac{d\widetilde{\Phi}_t(A)}{dt} = \widetilde{L}[\widetilde{\Phi}_t(A)], \tag{2.1}$$

in the Schrödinger and Heisenberg picture, respectively. Here, $\Phi_t(\widetilde{\Phi}_t)$ is the dynamical semigroup describing the irreversible time evolution of the open system in the Schrödinger (Heisenberg) representation; ρ is the density operator and A any operator acting on the Hilbert space $\mathbf{H}(A \in \mathbf{B}(\mathbf{H}))$; finally, $L(\widetilde{L})$ is the infinitesimal generator of the dynamical semigroup $\Phi_t(\widetilde{\Phi}_t)$.

By using the Lindblad theorem [7-9] which gives the most general form taken by the generator $\widetilde{L}$, the Markovian master equation (2.1) takes the form:

$$\frac{d\widetilde{\Phi}_t(A)}{dt} = \widetilde{L}(\widetilde{\Phi}_t(A)) = \frac{i}{\hbar}[H, \widetilde{\Phi}_t(A)] + \frac{1}{2\hbar}\sum_j (V_j^+[\widetilde{\Phi}_t(A), V_j] + [V_j^+, \widetilde{\Phi}_t(A)]V_j). \tag{2.2}$$

Here, H is the Hamiltonian of the system and the operators V_j, V_j^+ are taken as polynomials of only first degree in the observables $q_1, q_2, q_3, p_1, p_2, p_3$, which represent the hermitian generators of the Heisenberg group. Then in the linear space spanned by $q_k, p_k (k = 1, 2, 3)$, there exist only six linearly independent operators $V_{j=1,2,...,6}$:

$$V_j = \sum_{k=1}^{3} a_{jk} p_k + \sum_{k=1}^{3} b_{jk} q_k,$$

where $a_{jk}, b_{jk} \in \mathbf{C}$ with $j = 1, 2, ..., 6$.

Then it yields

$$V_j^+ = \sum_{k=1}^{3} a_{jk}^* p_k + \sum_{k=1}^{3} b_{jk}^* q_k,$$

where a_{jk}^*, b_{jk}^* are the complex conjugates of a_{jk}, b_{jk}.

The coordinates q_k and the momenta p_k obey the usual commutation relations $(k, l = 1, 2, 3)$:

$$[q_k, p_l] = i\hbar\delta_{kl}, \quad [q_k, q_l] = [p_k, p_l] = 0.$$

Inserting the operators V_j and V_j^+ into (2.2) we obtain:

$$\widetilde{L}(A) = \frac{i}{\hbar}[H, A] + \frac{1}{2\hbar}\sum_{k,l}\{(\frac{2}{\hbar}D_{q_k q_l} - i\alpha_{kl})(p_k[A, p_l] - [A, p_k]p_l) +$$

$$+(\frac{2}{\hbar}D_{p_kp_l}-i\beta_{kl})(q_k[A,q_l]-[A,q_k]q_l)-\frac{2}{\hbar}D_{q_kp_l}(p_k[A,q_l]-[A,p_k]q_l+q_l[A,p_k]-[A,q_l]p_k)-$$
$$-i\lambda_{kl}(p_k[A,q_l]-[A,p_k]q_l-q_l[A,p_k]+[A,q_l]p_k)\}.$$

Here we used the following abbreviations:

$$D_{q_kq_l}=D_{q_lq_k}=\frac{\hbar}{2}Re(\mathbf{a}_k^*\cdot\mathbf{a}_l),$$

$$D_{p_kp_l}=D_{p_lp_k}=\frac{\hbar}{2}Re(\mathbf{b}_k^*\cdot\mathbf{b}_l),$$

$$D_{q_kp_l}=D_{p_lq_k}=-\frac{\hbar}{2}Re(\mathbf{a}_k^*\cdot\mathbf{b}_l), \tag{2.3}$$

$$\alpha_{kl}=-\alpha_{lk}=-Im(\mathbf{a}_k^*\cdot\mathbf{a}_l),$$

$$\beta_{kl}=-\beta_{lk}=-Im(\mathbf{b}_k^*\cdot\mathbf{b}_l),$$

$$\lambda_{kl}=-Im(\mathbf{a}_k^*\cdot\mathbf{b}_l).$$

The scalar products are formed with the vectors $\mathbf{a}_k,\mathbf{b}_k$ and their complex conjugates $\mathbf{a}_k^*,\mathbf{b}_k^*$. The vectors have the components

$$\mathbf{a}_k=(a_{1k},a_{2k},...,a_{6k}),$$

$$\mathbf{b}_k=(b_{1k},b_{2k},...,b_{6k}).$$

3. The projection of the angular momentum

The general Hamiltonian of three uncoupled oscillators is

$$H=\sum_{k=1}^{3}(\frac{1}{2m_k}p_k^2+\frac{m_k\omega_k^2}{2}q_k^2).$$

The time-dependent expectation values of self-adjoint operators A and B can be written with the density operator ρ, describing the initial state of the quantum system, as follows:

$$m_A(t)=\mathrm{Tr}(\rho\widetilde{\Phi}_t(A)),$$

and, respectively,

$$\sigma_{AB}(t)=\frac{1}{2}\mathrm{Tr}(\rho\widetilde{\Phi}_t(AB+BA)).$$

In the following we denote the vector with the six components $m_{q_i}(t),m_{p_i}(t),i=1,2,3$, by $\mathbf{m}(t)$ and the following 6×6 matrix by $\hat{\sigma}(t)(i,j=1,2,3)$:

$$\hat{\sigma}(t)=\begin{pmatrix}\sigma_{q_iq_j} & \sigma_{q_ip_j}\\ \sigma_{p_iq_j} & \sigma_{p_ip_j}\end{pmatrix}.$$

Then via direct calculation of $\widetilde{L}(q_k)$ and $\widetilde{L}(p_k)$ we obtain

$$\frac{d\mathbf{m}}{dt} = \hat{Y}\mathbf{m}, \tag{3.1}$$

where

$$\hat{Y} = \begin{pmatrix} -\lambda_{11} & -\lambda_{12} & -\lambda_{13} & \frac{1}{m_1} & -\alpha_{12} & -\alpha_{13} \\ -\lambda_{21} & -\lambda_{22} & -\lambda_{23} & -\alpha_{21} & \frac{1}{m_2} & -\alpha_{23} \\ -\lambda_{31} & -\lambda_{32} & -\lambda_{33} & -\alpha_{31} & -\alpha_{32} & \frac{1}{m_3} \\ -m_1\omega_1^2 & \beta_{12} & \beta_{13} & -\lambda_{11} & -\lambda_{21} & -\lambda_{31} \\ \beta_{21} & -m_2\omega_2^2 & \beta_{23} & -\lambda_{12} & -\lambda_{22} & -\lambda_{32} \\ \beta_{31} & \beta_{32} & -m_3\omega_3^2 & -\lambda_{13} & -\lambda_{23} & -\lambda_{33} \end{pmatrix}. \tag{3.2}$$

From (3.1) it follows that

$$\mathbf{m}(t) = \hat{M}(t)\mathbf{m}(0) = \exp(t\hat{Y})\mathbf{m}(0), \tag{3.3}$$

where $\mathbf{m}(0)$ is given by the initial conditions. The matrix $\hat{M}(t)$ has to fulfil the condition

$$\lim_{t\to\infty} \hat{M}(t) = 0. \tag{3.4}$$

In order that this limit exists, $\hat{Y}$ must have only eigenvalues with negative real parts.

By direct calculation of $\widetilde{L}(q_k q_l), \widetilde{L}(p_k p_l)$ and $\widetilde{L}(q_k p_l + p_l q_k), (k, l = 1, 2, 3)$, we obtain

$$\frac{d\hat{\sigma}}{dt} = \hat{Y}\hat{\sigma} + \hat{\sigma}\hat{Y}^T + 2\hat{D}, \tag{3.5}$$

where $\hat{D}$ is the matrix of the diffusion coefficients $(i, j = 1, 2, 3)$

$$\hat{D} = \begin{pmatrix} D_{q_i q_j} & D_{q_i p_j} \\ D_{p_i q_j} & D_{p_i p_j} \end{pmatrix}$$

and $\hat{Y}^T$ the transposed matrix of $\hat{Y}$. The time-dependent solution of (3.5) can be written as

$$\hat{\sigma}(t) = \hat{M}(t)(\hat{\sigma}(0) - \hat{\Sigma})\hat{M}^T(t) + \hat{\Sigma}, \tag{3.6}$$

where $\hat{M}(t)$ is defined in (3.3). The matrix $\hat{\Sigma}$ is time independent and solves the static problem (3.5) $(d\hat{\sigma}/dt = 0)$:

$$\hat{Y}\hat{\Sigma} + \hat{\Sigma}\hat{Y}^T + 2\hat{D} = 0. \tag{3.7}$$

Now we assume that the following limit exists for $t \to \infty$:

$$\hat{\sigma}(\infty) = \lim_{t\to\infty} \hat{\sigma}(t). \tag{3.8}$$

In that case it follows from (3.6) and (3.4):

$$\hat{\sigma}(\infty) = \hat{\Sigma}. \tag{3.9}$$

Inserting (3.9) into (3.6) we obtain the basic equations for our purposes:

$$\hat{\sigma}(t) = \hat{M}(t)(\hat{\sigma}(0) - \hat{\sigma}(\infty))\hat{M}^T(t) + \hat{\sigma}(\infty), \tag{3.10}$$

where

$$\hat{Y}\hat{\sigma}(\infty) + \hat{\sigma}(\infty)\hat{Y}^T = -2\hat{D}. \tag{3.11}$$

For the calculation of the matrix $\hat{M}(t)$ we must diagonalize the matrix $\hat{Y}$ by solving the corresponding secular equation, i.e. $\det(\hat{Y} - z\hat{I}) = 0$, where z is the eigenvalue and $\hat{I}$ is the unit matrix. According to (3.2) one obtains an equation of sixth order for the eigenvalues z, which can be simply solved only for special examples. In the particular case with $\alpha_{kl} = 0, \beta_{kl} = 0, \lambda_{kl} = 0 (k \neq l)$, the secular equation is obtained as

$$[(z + \lambda_{11})^2 + \omega_1^2][(z + \lambda_{22})^2 + \omega_2^2][(z + \lambda_{33})^2 + \omega_3^2] = 0.$$

The roots of this equation are

$$z_{1,4} = -\lambda_{11} \pm i\omega_1, z_{2,5} = -\lambda_{22} \pm i\omega_2, z_{3,6} = -\lambda_{33} \pm i\omega_3. \tag{3.12}$$

Only positive values of $\lambda_{11}, \lambda_{22}, \lambda_{33}$ fulfil (3.4). Applying the eigenvalues z_i of $\hat{Y}$ we can write the time-dependent matrix $\hat{M}(t)$ as follows:

$$M_{mn}(t) = \sum_i N_{mi} \exp(z_i t) N_{in}^{-1},$$

where the matrix $\hat{N}$ represents the eigenvectors of $\hat{Y}$:

$$\sum_n Y_{mn} N_{ni} = z_i N_{mi}.$$

With the relations $M_{mn}(t = 0) = \delta_{mn}$ and $dM_{mn}(t)/dt|_{t=0} = Y_{mn}$ and using (3.3),(3.10) we conclude that the expectation values of the coordinates and momenta decay with the exponential factors $\exp(-\lambda_{11}t)$,$\exp(-\lambda_{22}t)$ and $\exp(-\lambda_{33}t)$ and the matrix elements σ_{mn} with the combined factors $\exp(-2\lambda_{11}t)$, $\exp(-\lambda_{22}t)$,$\exp(-\lambda_{33}t)$,$\exp[-(\lambda_{11} + \lambda_{22})t]$, $\exp[-(\lambda_{11} + \lambda_{33})t]$ and $\exp[-(\lambda_{22} + \lambda_{33})t]$.

We present here the matrix $\hat{M}(t)$ only for our special and simple case that the oscillators are uncoupled. With the roots given in (3.12) we obtain

$$\hat{M}(t) = \begin{pmatrix} M_1 & 0 & 0 & M_{11} & 0 & 0 \\ 0 & M_2 & 0 & 0 & M_{12} & 0 \\ 0 & 0 & M_3 & 0 & 0 & M_{13} \\ M_{21} & 0 & 0 & M_1 & 0 & 0 \\ 0 & M_{22} & 0 & 0 & M_2 & 0 \\ 0 & 0 & M_{23} & 0 & 0 & M_3 \end{pmatrix}, \tag{3.13}$$

where $(k = 1, 2, 3)$

$$M_k = \exp(-\lambda_{kk}t)\cos\omega_k t,$$

$$M_{1k}=\frac{1}{m_k\omega_k}\exp(-\lambda_{kk}t)\sin\omega_k t, \tag{3.14}$$

$$M_{2k}=-m_k\omega_k\exp(-\lambda_{kk}t)\sin\omega_k t.$$

This matrix can be used to evaluate $\hat\sigma(t)$ defined by (3.10). Since we are interested to obtain the expectation value of the projection of the angular momentum, we write down the following expressions for $\sigma_{q_1p_2}$ and $\sigma_{q_2p_1}$ with $\hat M(t)$ of (3.13),(3.14):

$$\sigma_{q_1p_2}(t)=\exp[-(\lambda_{11}+\lambda_{22})t]((\sigma_{q_1p_2}(0)-\sigma_{q_1p_2}(\infty))\cos\omega_1t\cos\omega_2t+$$

$$+\frac{1}{m_1\omega_1}(\sigma_{p_1p_2}(0)-\sigma_{p_1p_2}(\infty))\sin\omega_1t\cos\omega_2t-m_2\omega_2(\sigma_{q_1q_2}(0)-\sigma_{q_1q_2}(\infty))\cos\omega_1t\sin\omega_2t-$$

$$-\frac{m_2\omega_2}{m_1\omega_1}(\sigma_{q_2p_1}(0)-\sigma_{q_2p_1}(\infty))\sin\omega_1t\sin\omega_2t)+\sigma_{q_1p_2}(\infty), \tag{3.15}$$

$$\sigma_{q_2p_1}(t)=\exp[-(\lambda_{11}+\lambda_{22})t]((\sigma_{q_2p_1}(0)-\sigma_{q_2p_1}(\infty))\cos\omega_1t\cos\omega_2t-$$

$$-m_1\omega_1(\sigma_{q_1q_2}(0)-\sigma_{q_1q_2}(\infty))\sin\omega_1t\cos\omega_2t+\frac{1}{m_2\omega_2}(\sigma_{p_1p_2}(0)-\sigma_{p_1p_2}(\infty))\cos\omega_1t\sin\omega_2t-$$

$$-\frac{m_1\omega_1}{m_2\omega_2}(\sigma_{q_1p_2}(0)-\sigma_{q_1p_2}(\infty))\sin\omega_1t\sin\omega_2t)+\sigma_{q_2p_1}(\infty). \tag{3.16}$$

Similar expressions are found for the other matrix elements of $\hat\sigma(t)$. The matrix elements of $\hat\sigma(\infty)$ depend on $\hat Y$ and $\hat D$ and must be evaluated with (3.11) or by the relation [11]:

$$\hat\sigma(\infty)=2\int_0^\infty\hat M(t')\hat D\hat M^T(t')dt'.$$

We obtain that

$$\sigma_{q_1p_2}(\infty)=\sigma_{q_2p_1}(\infty). \tag{3.17}$$

The expectation value of the projection of the angular momentum can be written from (3.15) and (3.16):

$$<L_3(t)>=\sigma_{q_1p_2}(t)-\sigma_{q_2p_1}(t). \tag{3.18}$$

If the three uncoupled oscillators have the same mass and frequency, then we obtain

$$<L_3(t)>=(\sigma_{q_1p_2}(0)-\sigma_{q_2p_1}(0))\exp[-(\lambda_{11}+\lambda_{22})t]. \tag{3.19}$$

If we consider the open system to be symmetric, then from (2.3)we have $\lambda_{11}=\lambda_{22}=\lambda$ and (3.19) becomes:

$$<L_3(t)>=<L_3(0)>\exp(-2\lambda t),$$

with $<L_3(t)>|_{t\to\infty}\to 0$ if $\lambda>0$. This evolution law for the projection of the angular momentum is identical to that obtained in [14] by another method. We should like to mention that the same evolution law for the projection $<L_3(t)>$ of the angular momentum can also be obtained for only two uncoupled harmonic oscillators.

4. The squared angular momentum

In this Section we should like to discuss the behaviour of the squared angular momentum $< L^2(t) >$ for a system of three independent damped harmonic oscillators. In this case the expectation value of the angular momentum can be obtained directly from the following expression:

$$< L^2(t) >=< p_1^2(t) >< q_2^2(t) > + < p_1^2(t) >< q_3^2(t) > + < p_2^2(t) >< q_1^2(t) > +$$

$$+ < p_2^2(t) >< q_3^2(t) > + < p_3^2(t) >< q_1^2(t) > + < p_3^2(t) >< q_2^2(t) > -$$

$$-\frac{1}{2} < (p_1q_1+q_1p_1)(t) >< (p_2q_2+q_2p_2)(t) > -\frac{1}{2} < (p_2q_2+q_2p_2)(t) >< (p_3q_3+q_3p_3)(t) > -$$

$$-\frac{1}{2} < (p_3q_3 + q_3p_3)(t) >< (p_1q_1 + q_1p_1)(t) > -\frac{3}{2}\hbar^2. \tag{4.1}$$

The expectation values in (4.1) can be obtained by using the expressions for the centroids and variances for the one-dimensional harmonic oscillator obtained in [10]. We have ($k = 1, 2, 3$):

$$\begin{aligned} &< p_k^2(t) >= \sigma_{(pp)_k}(t) + \sigma_{p_k}^2(t), \\ &< q_k^2(t) >= \sigma_{(qq)_k}(t) + \sigma_{q_k}^2(t), \\ \frac{1}{2} &< (p_kq_k + q_kp_k)(t) >= \sigma_{(pq)_k}(t) + \sigma_{p_k}(t)\sigma_{q_k}(t). \end{aligned} \tag{4.2}$$

For each harmonic oscillator the Hamiltonian is chosen of the form (k=1,2,3)

$$H_k = \frac{1}{2m_k}p_k^2 + \frac{m_k\omega_k^2}{2}q_k^2 + \frac{\mu_k}{2}(p_kq_k + q_kp_k)$$

and the operators in the Lindblad generator can be written in the form

$$V_i^{(k)} = a_i^{(k)}p_k + b_i^{(k)}q_k, i = 1, 2,$$

with $a_i^{(k)}, b_i^{(k)}$ complex numbers.

In the following we consider, for simplicity, the system of three harmonic oscillators with the same mass m, the same frequency ω and the same opening coefficients D_{qq}, D_{pp}, D_{pq} and λ. In addition, we take $\mu = 0$. Then, by introducing the expressions (4.2) taken from [10] into (4.1), we obtain, after long, but straightforward calculations, the following form for the expectation value of the squared angular momentum:

$$< L^2(t) >= \beta e^{-4\lambda t}+$$

$$+e^{-2\lambda t}\{\beta_1 \sin^2 \omega t + \beta_2 \cos^2 \omega t + \beta_3 \sin \omega t + \beta_4 \cos \omega t + \beta_5 \sin \omega t \cos \omega t\}+$$

$$+L^2(\infty),$$

where $\beta_j (j = 1, ..., 5)$ are constants. The asymptotic value (if $\lambda > 0$) is given by

$$< L^2(\infty) >= \sigma_{(pp)_1}(\infty)\sigma_{(qq)_2}(\infty) + \sigma_{(qq)_1}(\infty)\sigma_{(pp)_2}(\infty) - 2\sigma_{(pq)_1}(\infty)\sigma_{(pq)_2}(\infty) + (cyclic) -$$

$$-\frac{3}{2}\hbar^2 = 6(\sigma_{pp}(\infty)\sigma_{qq}(\infty) - \sigma_{pq}^2(\infty)) - \frac{3}{2}\hbar^2,$$

where [10]

$$\sigma_{qq}(\infty) = \frac{1}{2(m\omega)^2\lambda(\lambda^2 + \omega^2)}((m\omega)^2(2\lambda^2 + \omega^2)D_{qq} + \omega^2 D_{pp} + 2m\omega^2\lambda D_{pq}),$$

$$\sigma_{pp}(\infty) = \frac{1}{2\lambda(\lambda^2 + \omega^2)}((m\omega)^2\omega^2 D_{qq} + (2\lambda^2 + \omega^2)D_{pp} - 2m\omega^2\lambda D_{pq}),$$

$$\sigma_{pq}(\infty) = \frac{1}{2m\lambda(\lambda^2 + \omega^2)}(-\lambda(m\omega)^2 D_{qq} + \lambda D_{pp} + 2m\lambda^2 D_{pq}).$$

If $\lambda > 0$, this evolution law shows an exponential damping of the angular momentum of the system of three harmonic oscillators, like in [14].

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